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Laguerre–Samuelson type inequalities

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Abstract

We give a generalization of Laguerre–Samuelson's inequality in the framework of group theory. This extends a recent result of Trenkler and Puntanen [G. Trenkler, S. Puntanen, A multivariate version of Samuelson's inequality, Linear Algebra Appl. 410 (2005) 143–149]. Some examples are given.

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1. Introduction

The Laguerre–Samuelson inequality [3,7] asserts that for any real numbers $x_1, x_2, \ldots, x_n$

$$(x_j - \overline{x})^2 \leqslant (n-1)s^2, \tag{1}$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^n x_i$ and $s^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2$. Recently, Trenkler and Puntanen [8, Theorem 1] proved the following generalization of (1):

$$(n-1)\mathbf{S} - (\mathbf{x}_j - \overline{\mathbf{x}})(\mathbf{x}_j - \overline{\mathbf{x}})^{\mathrm{T}} \text{ is nonnegative definite,} \tag{2}$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are vectors in $\mathbb{R}^p$ (and represent a sample from p-dimensional distribution), $\overline{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i$ and $\mathbf{S} = \frac{1}{n}\sum_{i=1}^n (\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})^{\mathrm{T}}$. Observe that one can obtain inequality (1) by looking at the $(1, 1)$ entry of (2).

In the present paper we study inequalities like (1) and (2), where the additive group $(\mathbb{R}^p, +)$ and the operation of arithmetic mean of vectors is replaced by an abelian group endowed with

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mean operation. Moreover, in our results the role of the quadratic function in (1) and (2) is played by a map φ which is "semiconvex" with respect to the mean operations. This gives a number of generalizations of (1) and (2) for both multiplicative and additive groups.

In Section 2 we present a general approach to this problem. We show that L–S type inequalities are closely related to the notion of semiconvexity (see Theorem 2.1). Section 3 is devoted to inequalities induced by additive groups of linear spaces. In particular, we present results (see Corollaries 3.1 and 3.4) extending the above-mentioned result of Trenkler and Puntanen (see (2)). In addition, some examples are given to illustrate the theory.

Throughout the paper $\mathbb{R}^p$ denotes the Euclidean space of real column p-tuples, $\mathbb{M}_p$ (resp. $\mathbb{H}_p$) is the space of real $p \times p$ matrices (resp. $p \times p$ Hermitian matrices), and $\mathbb{L}_p$ stands for the Loewner cone of real $p \times p$ nonnegative definite matrices.

2. Results

We say that an abelian semigroup $(G, \cdot)$ is endowed with *mean operation*, if for any $a_1, \dots, a_n \in G$ there exists a unique element $b = (a_1 \cdots a_n)^{\frac{1}{n}} \in G$ such that $b^n = a_1 \cdots a_n$. As usual, the symbol b^n denotes the product $b \cdots b$ with n factors b. A relation $\preceq$ on a set H is said to be a *preorder* if (1) $\mathbf{x} \preceq \mathbf{x}$ for $\mathbf{x} \in H$ (reflexivity), and (2) $\mathbf{x} \preceq \mathbf{y}$, $\mathbf{y} \preceq \mathbf{z}$ implies $\mathbf{x} \preceq \mathbf{z}$ for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in H$ (transitivity).

Given two abelian semigroups $(G, \cdot)$ and $(H, \cdot)$ endowed with their mean operations, and given a preorder $\preceq$ on H, we say that a map $\varphi\colon G \to H$ is *semiconvex in* G,H-means (in short, G,H-semiconvex) if for every positive integer n

$$\varphi\left((a_1 \cdots a_n)^{\frac{1}{n}}\right) \preceq (\varphi(a_1) \cdots \varphi(a_n))^{\frac{1}{n}} \quad \text{for any } a_1, \dots, a_n \in G \tag{3}$$

(cf. [5]; see (17) for additive counterpart of (3)).

Let $\varphi\colon G \to H$ be a map, and let k_n, $n = 1, 2, \dots$ be a positive rational sequence. If for every positive integer n

$$\varphi\left(a^{\frac{1}{n}}\right) = \left([\varphi(a)]^{k_n}\right)^{\frac{1}{n}} \quad \text{for } a \in G, \tag{4}$$

then φ is called k_n*-semihomogeneous* map. (See (16) for additive version of (4).) If for every n

$$\left([\varphi(a)]^{k_n}\right)^{\frac{1}{n}} \preceq \varphi\left(a^{\frac{1}{n}}\right) \quad \text{for } a \in G, \tag{5}$$

then φ is said to be *weakly* k_n*-semihomogeneous*.

We say that a preorder $\preceq$ on H *preserves algebraic operations* if

$$b \preceq a \text{ implies } bc \preceq ac \;\text{ and }\; b^n \preceq a^n \;\text{ and }\; b^{\frac{1}{n}} \preceq a^{\frac{1}{n}} \quad \text{for } a, b, c \in H, n \in \mathbb{N}. \tag{6}$$

If G is a group, then a map φ defined on G is said to be *even* if $\varphi\left(a^{-1}\right) = \varphi(a)$ for $a \in G$.

Theorem 2.1. *Let $(G, \cdot)$ and $(H, \cdot)$ be abelian groups. Suppose that G and H are endowed with mean operations. Let $\varphi\colon G \to H$ be an even map. Assume $\preceq$ is a preorder on H preserving algebraic operations. Let k_n, $n = 1, 2, \dots$ be a sequence of positive rational numbers.*
If φ is k_n-semihomogeneous then the following two statements are equivalent:

(i) *The map φ is G, H-semiconvex.*
(ii) *Laguerre–Samuelson (L–S) type inequality holds, that is, for every positive integer n and every $j = 1, \dots, n$*

$$\left[\varphi\left(g_j\tilde{g}^{-1}\right)\right]^{k_{n-1}+1} \preceq \varphi\left(g_1\tilde{g}^{-1}\right)\cdots\varphi\left(g_n\tilde{g}^{-1}\right) \quad \text{for } g_1,\ldots,g_n \in G, \tag{7}$$

where $\tilde{g} = (g_1 \cdots g_n)^{\frac{1}{n}}$.

If φ *is weakly* k_n*-semihomogeneous then the implication* (i) $\Rightarrow$ (ii) *is valid.*

Proof. Fix any $g_1,\ldots,g_n \in G$. Denote $a_i = g_i\tilde{g}^{-1}$ for $i = 1,\ldots,n$. Because G is abelian, we obtain $a_1\cdots a_n = e$, where e is the unit of G. Hence $a_j = (a_1\cdots a_{j-1}a_{j+1}\cdots a_n)^{-1}$, and therefore

$$\varphi(a_j) = \varphi(a_1\cdots a_{j-1}a_{j+1}\cdots a_n), \tag{8}$$

since φ is even. Using (4) and (8) leads to

$$\begin{aligned}\varphi\left((a_1\cdots a_{j-1}a_{j+1}\cdots a_n)^{\frac{1}{n-1}}\right) &= \left([\varphi(a_1\cdots a_{j-1}a_{j+1}\cdots a_n)]^{k_{n-1}}\right)^{\frac{1}{n-1}}\\ &= \left([\varphi(a_j)]^{k_{n-1}}\right)^{\frac{1}{n-1}}.\end{aligned} \tag{9}$$

(i) $\Rightarrow$ (ii). From (3) we obtain

$$\varphi\left((a_1\cdots a_{j-1}a_{j+1}\cdots a_n)^{\frac{1}{n-1}}\right) \preceq \left[\varphi(a_1)\cdots\varphi(a_{j-1})\varphi(a_{j+1})\cdots\varphi(a_n)\right]^{\frac{1}{n-1}}. \tag{10}$$

Combining (10) and (9) we get

$$\left([\varphi(a_j)]^{k_{n-1}}\right)^{\frac{1}{n-1}} \preceq \left[\varphi(a_1)\cdots\varphi(a_{j-1})\varphi(a_{j+1})\cdots\varphi(a_n)\right]^{\frac{1}{n-1}},$$

which implies, by (6),

$$[\varphi(a_j)]^{k_{n-1}} \preceq \varphi(a_1)\cdots\varphi(a_{j-1})\varphi(a_{j+1})\cdots\varphi(a_n).$$

Multiplying both sides of the last inequality by $\varphi(a_j)$ yields

$$[\varphi(a_j)]^{k_{n-1}+1} \preceq \varphi(a_1)\cdots\varphi(a_n),$$

since H is abelian. This proves (7), as required.

(ii) $\Rightarrow$ (i). To see this implication, backtrack the steps above and apply (6).

The proof of the last part of the theorem is similar to that of (i) $\Rightarrow$ (ii). $\square$

Remark 2.2

(1) In the proof of the implication (i) $\Rightarrow$ (ii) of Theorem 2.1, the last part of (6), i.e.,

$$b \preceq a \text{ implies } b^{\frac{1}{n}} \preceq a^{\frac{1}{n}} \quad \text{for } a, b \in H$$

is superfluous. It plays a role in the proof of (ii) $\Rightarrow$ (i).

(2) It is sufficient to assume that H is semigroup to prove (i) $\Rightarrow$ (ii).

(3) If H is not abelian, then (7) takes the form

$$\begin{aligned}\left[\varphi\left(g_j\tilde{g}^{-1}\right)\right]^{k_{n-1}+1} \preceq\ & \varphi\left(g_1\tilde{g}^{-1}\right)\cdots\varphi\left(g_{j-1}\tilde{g}^{-1}\right)\varphi\left(g_{j+1}\tilde{g}^{-1}\right)\\ &\cdots\varphi\left(g_n\tilde{g}^{-1}\right)\varphi\left(g_j\tilde{g}^{-1}\right) \quad \text{for } g_1,\ldots,g_n \in G.\end{aligned}$$

We now show how Trenkler–Puntanen's result [8, Theorem 1] (see (2)) can be deduced from our Theorem 2.1.

Consider the additive groups $G = (\mathbb{R}^p, +)$ and $H = (\mathbb{M}_p, +)$ endowed with the arithmetic mean operations

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \quad \text{for } \mathbf{x}_i \in \mathbb{R}^p$$

and

$$\bar{\mathbf{y}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{y}_i \quad \text{for } \mathbf{y}_i \in \mathbb{M}_p.$$

Let $\leqslant$ be the Loewner order on $\mathbb{M}_p$ defined by: $\mathbf{y} \leqslant \mathbf{z}$ iff $\mathbf{z} - \mathbf{y}$ is nonnegative definite. Take $\varphi(\mathbf{x}) = \mathbf{x}\mathbf{x}^{\mathrm{T}}$ for $\mathbf{x} \in \mathbb{R}^p$. By direct computations, one obtains that the map φ is semiconvex in the sense that

$$\varphi\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i\right) \leqslant \frac{1}{n}\sum_{i=1}^{n} \varphi(\mathbf{x}_i) \quad \text{for } \mathbf{x}_i \in \mathbb{R}^p.$$

Moreover, (4) is met for $k_n = \frac{1}{n}$, because $\varphi\left(\frac{1}{n}\mathbf{x}\right) = \frac{1}{n^2}\varphi(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^p$. So, $k_{n-1} + 1 = \frac{n}{n-1}$. In consequence, (7) reduces to the mentioned result of Trenkler and Puntanen.

We now present further interpretations of Theorem 2.1.

Example 2.3 (*Multiplicative–additive L–S type inequality*). Put $G = (\mathbb{R}_+, \cdot)$ and $H = (\mathbb{R}, +)$. Let $\preceq$ be the usual order $\leqslant$ on $\mathbb{R}$. Consider a map $\varphi : \mathbb{R}_+ \to \mathbb{R}$, which is G,H-semiconvex, even and k_n-semihomogeneous, i.e.,

$$\varphi\left(\sqrt[n]{a_1 \cdots a_n}\right) \leqslant \frac{\varphi(a_1) + \cdots + \varphi(a_n)}{n} \quad \text{for } a_1, \ldots, a_n \in \mathbb{R}_+, \tag{11}$$

$$\varphi\left(a^{-1}\right) = \varphi(a) \quad \text{and} \quad \varphi(a^{\frac{1}{n}}) = \frac{1}{n}k_n\varphi(a) \quad \text{for } a \in \mathbb{R}_+. \tag{12}$$

For example, (11) and (12) are satisfied for the map $\varphi(t) = |\ln t|$ for $t \in \mathbb{R}_+$ with $k_n = 1$.

Clearly, condition (6) is fulfilled. Using (7) we obtain the inequality

$$(k_{n-1} + 1)\varphi\left(\frac{x_j}{\sqrt[n]{x_1 \cdots x_n}}\right) \leqslant \sum_{i=1}^{n} \varphi\left(\frac{x_i}{\sqrt[n]{x_1 \cdots x_n}}\right) \quad \text{for } x_1, \ldots, x_n \in \mathbb{R}_+.$$

Example 2.4 (*Multiplicative L–S type inequality*). Put $G = H = (\mathbb{R}_+, \cdot)$. Let $\preceq$ be the usual order $\leqslant$ on $\mathbb{R}_+$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be G, H-semiconvex, even and k_n-semihomogeneous map, i.e.,

$$\varphi\left(\sqrt[n]{a_1 \cdots a_n}\right) \leqslant \sqrt[n]{\varphi(a_1) \cdots \varphi(a_n)} \quad \text{for } a_1, \ldots, a_n \in \mathbb{R}_+, \tag{13}$$

$$\varphi\left(a^{-1}\right) = \varphi(a) \quad \text{and} \quad \varphi\left(a^{\frac{1}{n}}\right) = \left[(\varphi(a))^{k_n}\right]^{\frac{1}{n}} \quad \text{for } a \in \mathbb{R}_+. \tag{14}$$

As noted in [5, pp. 158–159], (13) is fulfilled for each polynomial (and real analytical function) with nonnegative coefficients. For instance, conditions (13) and (14) are satisfied for the map $\varphi(t) = \mathrm{e}^{(\ln t)^4}$ for $t \in \mathbb{R}_+$ with $k_n = \frac{1}{n^3}$.

It follows from (7) that

$$\left[\varphi\left(\frac{x_j}{\sqrt[n]{x_1\cdots x_n}}\right)\right]^{k_{n-1}+1} \leqslant \prod_{i=1}^{n}\varphi\left(\frac{x_i}{\sqrt[n]{x_1\cdots x_n}}\right) \quad \text{for } x_1,\ldots,x_n \in \mathbb{R}_+.$$

Example 2.5 (*Additive–multiplicative L–S type inequality*). Let $G = (\mathbb{H}_p, +)$ be the additive group of the linear space of all Hermitian $p \times p$ matrices, and let $H = (\mathbb{R}_+^p, \circ)$ be the abelian group of all positive p-tuples provided with Schur (entrywise) product $\circ$. For $\alpha \in \mathbb{R}$ and $\mathbf{a} = (a_1,\ldots,a_p)^{\mathrm{T}} \in \mathbb{R}_+^p$ we denote $\mathbf{a}^\alpha = (a_1^\alpha,\ldots,a_p^\alpha)^{\mathrm{T}} \in \mathbb{R}_+^p$.

We equip $\mathbb{R}_+^p$ with the weak majorization order $\preceq_w$. That is, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$ we write $\mathbf{b} \preceq_w \mathbf{a}$ and say that $\mathbf{a}$ *weakly majorizes* $\mathbf{b}$ if for each $k = 1,\ldots,p$ the sum of k largest entries of $\mathbf{b}$ does not exceed the sum of k largest entries of $\mathbf{a}$. By $\lambda(\mathbf{A}) = \big(\lambda_1(\mathbf{A}),\ldots,\lambda_p(\mathbf{A})\big)^{\mathrm{T}}$ we denote the vector of eigenvalues of $\mathbf{A} \in \mathbb{H}_p$ arranged in nondecreasing order, that is $\lambda_1(\mathbf{A}) \geqslant \cdots \geqslant \lambda_p(\mathbf{A})$. The symbol $\sigma(\mathbf{A})$ stands for the vector of singular values $\sigma_1(\mathbf{A}) \geqslant \cdots \geqslant \sigma_p(\mathbf{A})$. If $\mathbf{A}, \mathbf{B} \in \mathbb{H}_p$ are positive definite then

$$\lambda(\mathbf{AB}) = |\lambda(\mathbf{AB})| \preceq_w \sigma(\mathbf{AB}) \preceq_w \sigma(\mathbf{A}) \circ \sigma(\mathbf{B}) = \lambda(\mathbf{A}) \circ \lambda(\mathbf{B}). \tag{15}$$

In fact, the first equality holds because the product of positive definite matrices has nonnegative eigenvalues [9, p. 1174]. The first inequality is due to Weyl's Majorant Theorem (see [2, Theorem II.3.6]). The second one is a consequence of [2, Theorem IV.2.5]. The last equality is clear, since $\mathbf{A}$ and $\mathbf{B}$ are positive definite.

As usual, if $f : \mathbb{R} \to \mathbb{R}$ and $\mathbf{A} = U \operatorname{diag}\big(\lambda_1(\mathbf{A}),\ldots,\lambda_p(\mathbf{A})\big)U^*$ with unitary $p \times p$ matrix U, then $f(\mathbf{A})$ is defined by $f(\mathbf{A}) = U \operatorname{diag}\big(f(\lambda_1(\mathbf{A})),\ldots,f(\lambda_p(\mathbf{A}))\big)U^*$ (see [1, p. 219]). Thus $\lambda f(\mathbf{A}) = \big(f(\lambda_1(\mathbf{A})),\ldots,f(\lambda_p(\mathbf{A}))\big)^{\mathrm{T}}$.

Let f be a (positive) log-convex function on $\mathbb{R}$. Aujla and Silva [1, Theorem 3.4] showed that for Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$

$$\lambda f(\alpha\mathbf{A} + (1-\alpha)\mathbf{B}) \preceq_w \lambda\big(f(\mathbf{A})^\alpha f(\mathbf{B})^{1-\alpha}\big) \quad \text{for } \alpha \in [0,1].$$

Combining this result and (15) we get

$$\lambda f(\alpha\mathbf{A} + (1-\alpha)\mathbf{B}) \preceq_w \lambda\big(f(\mathbf{A})^\alpha\big) \circ \lambda\big(f(\mathbf{B})^{1-\alpha}\big) \quad \text{for } \alpha \in [0,1].$$

Since $f^{1-\alpha}$ is log-convex, it follows from the above and from [1, Lemma 2.14] that the function $\varphi : \mathbb{H}_p \to \mathbb{R}_+^p$ defined by $\varphi(\mathbf{A}) = \lambda f(\mathbf{A})$ is G,H-semiconvex with respect to $\preceq_w$.

If, in addition, f is even and k_n-semihomogeneous, then by virtue of Theorem 2.1, for $j = 1,\ldots,n$

$$\Big(\lambda f(\mathbf{X}_j - \overline{\mathbf{X}})\Big)^{k_{n-1}+1} \preceq_w \lambda f(\mathbf{X}_1 - \overline{\mathbf{X}}) \circ \cdots \circ \lambda f(\mathbf{X}_n - \overline{\mathbf{X}}) \quad \text{for } \mathbf{X}_1,\ldots,\mathbf{X}_n \in \mathbb{H}_p,$$

where $\overline{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^n \mathbf{X}_i$. For instance, for $f(t) = \mathrm{e}^{t^2}$ with $k_n = \frac{1}{n}$, we obtain

$$\Big(\lambda \mathrm{e}^{(\mathbf{X}_j - \overline{\mathbf{X}})^2}\Big)^{\frac{n}{n-1}} \preceq_w \lambda \mathrm{e}^{(\mathbf{X}_1 - \overline{\mathbf{X}})^2} \circ \cdots \circ \lambda \mathrm{e}^{(\mathbf{X}_n - \overline{\mathbf{X}})^2}.$$

3. The additive case

In this section we exploit the additive groups of linear spaces. That is, given real linear spaces V and W, we employ the groups

$$(G,\cdot)=(V,+) \quad \text{and} \quad (H,\cdot)=(W,+),$$

where + denotes the operation of vector addition in V and W. Here we use the arithmetic mean operations

$$\overline{\mathbf{x}}=\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i \quad \text{and} \quad \overline{\mathbf{y}}=\frac{1}{n}\sum_{i=1}^{n}\mathbf{y}_i,$$

where $\mathbf{x}_1,\ldots,\mathbf{x}_n$ are vectors in V and $\mathbf{y}_1,\ldots,\mathbf{y}_n$ are vectors in W.

A (nonempty) subset $C\subset W$ is called a *convex cone* if $\alpha\mathbf{x}+\beta\mathbf{y}\in C$ for all vectors $\mathbf{x},\mathbf{y}\in C$ and nonnegative scalars $\alpha,\beta\in\mathbb{R}$. The *cone preorder* $\preceq_C$ induced by a convex cone C is defined as follows: for $\mathbf{x},\mathbf{y}\in W$

$$\mathbf{y}\preceq_C\mathbf{x} \quad \text{iff} \quad \mathbf{x}-\mathbf{y}\in C.$$

It is easily seen that for $\mathbf{x},\mathbf{y},\mathbf{z}\in W$ and $n=1,2,\ldots,$

$$\mathbf{y}\preceq_C\mathbf{x} \quad \text{implies} \quad \mathbf{y}+\mathbf{z}\preceq_C\mathbf{x}+\mathbf{z},\ \ n\mathbf{y}\preceq_C n\mathbf{x} \ \text{and} \ \frac{1}{n}\mathbf{y}\preceq_C\frac{1}{n}\mathbf{x},$$

which is an additive version of (6).

Under the above assumptions, a map $\varphi: V\to W$ is even iff $\varphi(-\mathbf{x})=\varphi(\mathbf{x})$ for $\mathbf{x}\in V$. The requirement (4) takes the form as follows:

$$\varphi\left(\frac{1}{n}\mathbf{x}\right)=\frac{1}{n}k_n\varphi(\mathbf{x}) \quad \text{for } \mathbf{x}\in V \ (k_n\text{-semihomogeneity}), \tag{16}$$

where k_n, $n=1,2,\ldots,$ is a sequence of positive (rational) numbers.

Corollary 3.1. *Let* $(G,\cdot)=(V,+)$ *and* $(H,\cdot)=(W,+)$ *be the additive groups of real linear spaces* V *and* W, *respectively. Let* $C\subset W$ *be a convex cone inducing preorder* $\preceq_C$ *on* W. *Suppose that* $\varphi: V\to W$ *is an even map satisfying* (16).
Then the following two statements are equivalent.

(i) *The map* φ *is* G,H*-semiconvex, that is,*

$$\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\right)\preceq_C\frac{1}{n}\sum_{i=1}^{n}\varphi(\mathbf{x}_i) \quad \textit{for } \mathbf{x}_1,\ldots,\mathbf{x}_n\in V. \tag{17}$$

(ii) *L–S type inequality holds, that is, for* $j=1,\ldots,n$

$$(k_{n-1}+1)\varphi(\mathbf{x}_j-\overline{\mathbf{x}})\preceq_C\sum_{i=1}^{n}\varphi(\mathbf{x}_i-\overline{\mathbf{x}}) \quad \textit{for } \mathbf{x}_1,\ldots,\mathbf{x}_n\in V. \tag{18}$$

Proof. Use Theorem 2.1 in the setting described at the begining of Section 3. □

Example 3.2 (*L–S type inequality for singular values of matrices*). Put $V=\mathbb{M}_p$ and $W=\mathbb{R}^p$. Consider the weak majorization order $\preceq_w$ on $\mathbb{R}^p$. As usual, the vector of the singular values of a matrix $\mathbf{X}\in\mathbb{M}_p$ is denoted by $\sigma(\mathbf{X})$. It is known that (see [2, p. 35]) for $\mathbf{X},\mathbf{Y}\in\mathbb{M}_p$ and $t>0$

$$\sigma(t\mathbf{X})=t\sigma(\mathbf{X}) \quad \text{and} \quad \sigma(\mathbf{X}+\mathbf{Y})\preceq_w\sigma(\mathbf{X})+\sigma(\mathbf{Y}), \tag{19}$$

$$\sigma(\mathbf{Y})\preceq_w\sigma(\mathbf{X}) \quad \text{iff} \quad \sigma(\mathbf{Y})\preceq_C\sigma(\mathbf{X}), \tag{20}$$

where $D = \{\mathbf{x} = (x_1, \ldots, x_p)^{\mathrm{T}} \in \mathbb{R}^p : x_1 \geqslant \cdots \geqslant x_p \geqslant 0\}$ and C is the dual cone of D, i.e., $C = \text{dual } D = \{\mathbf{y} = (y_1, \ldots, y_p)^{\mathrm{T}} \in \mathbb{R}^p : \sum_{i=1}^p x_i y_i \geqslant 0 \quad \text{for all } \mathbf{x} \in D\}$ (see [6, Example 4.4]).

Let $\varphi : \mathbb{M}_p \to \mathbb{R}^p$ be the map defined by $\varphi(\mathbf{X}) = \sigma(|\mathbf{X}|^r)$ with $r \geqslant 2$, where $|\mathbf{X}| = (X^* X)^{\frac{1}{2}}$. It follows from [4, Theorem 3] and (19) that if $\alpha_i > 0$, $i = 1, \ldots, n$, with $\sum_{i=1}^n \alpha_i = 1$, then

$$\sigma\left(\left|\sum_{i=1}^n \alpha_i \mathbf{X}_i\right|^r\right) \leqslant_w \sigma\left(\sum_{i=1}^n \alpha_i |\mathbf{X}_i|^r\right) \leqslant_w \sum_{i=1}^n \alpha_i \sigma\left(|\mathbf{X}_i|^r\right). \tag{21}$$

Therefore conditions (16) and (17) are satisfied with $k_n = \frac{1}{n^{r-1}}$. In consequence, (18) reduces to

$$\left(\frac{1}{(n-1)^{r-1}} + 1\right)\sigma\left(|\mathbf{X}_j - \overline{\mathbf{X}}|^r\right) \preceq_w \sum_{i=1}^n \sigma\left(|\mathbf{X}_i - \overline{\mathbf{X}}|^r\right) \quad \text{for } \mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbb{M}_p, \tag{22}$$

where $\overline{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^n \mathbf{X}_i$.

We now illustrate Corollary 3.1 for $V = \mathbb{H}_p$ and $W = \mathbb{R}^p$ and for the weak majorization order $\preceq_w$ on $\mathbb{R}^p$.

Example 3.3 (*L–S type inequality for convex functions*). Aujla and Silva [1, Theorem 2.3] proved that if $f : (a, b) \to \mathbb{R}$ is a convex function, then

$$\lambda f(\alpha\mathbf{A} + (1-\alpha)\mathbf{B}) \preceq_w \lambda(\alpha f(\mathbf{A}) + (1-\alpha) f(\mathbf{B})) \quad \text{for } \alpha \in [0, 1] \text{ and } \mathbf{A}, \mathbf{B} \in H_p(I),$$

$H_p(I)$ being the set of all $p \times p$ Hermitian matrices having spectrum in $I = (a, b)$. Moreover, Ky Fan's maximum principle [2, p. 35] guarantees that

$$\lambda(\mathbf{A} + \mathbf{B}) \preceq_m \lambda(\mathbf{A}) + \lambda(\mathbf{B}) \quad \text{for } \mathbf{A}, \mathbf{B} \in \mathbb{H}_p.$$

It follows from the above that the function $\varphi(\mathbf{A}) = \lambda f(\mathbf{A})$, $\mathbf{A} \in \mathbb{H}_p(I)$, is semiconvex with respect to $\preceq_w$ in the sense of (17).

Assume that f is even, k_n-semihomogeneous and convex on $I = (-\infty, \infty)$. By virtue of Corollary 3.1 for $j = 1, \ldots, n$

$$(k_{n-1} + 1)\lambda f(\mathbf{X}_j - \overline{\mathbf{X}}) \preceq_w \sum_{i=1}^n \lambda f(\mathbf{X}_i - \overline{\mathbf{X}}) \quad \text{for } \mathbf{X}_1, \ldots, \mathbf{X}_n \in \mathbb{H}_p.$$

For instance, for $f(t) = t^2$ with $k_n = \frac{1}{n}$, we obtain

$$\frac{n}{n-1}\lambda(\mathbf{X}_j - \overline{\mathbf{X}})^2 \preceq_w \sum_{i=1}^n \lambda(\mathbf{X}_i - \overline{\mathbf{X}})^2.$$

Finally, applying Corollary 3.1 for $V = \mathbb{R}^p$, $W = \mathbb{M}_p$ and $C = \mathbb{L}_p$ we get

Corollary 3.4. *Let $\varphi : \mathbb{R}^p \to \mathbb{M}_p$ be an even map satisfying* (16).
Then the following two statements are equivalent.

(i) *The map φ is semiconvex in the sense of* (17) *with $\preceq_C$ replaced by $\preceq_{\mathbb{L}_p}$.*
(ii) *For every n, $j = 1, \ldots, n$ and $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^p$*

$$\sum_{i=1}^{n} \varphi(\mathbf{x}_i - \overline{\mathbf{x}}) - (k_{n-1} + 1)\varphi(\mathbf{x}_j - \overline{\mathbf{x}}) \quad \textit{is nonnegative definite.} \tag{23}$$

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